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On Two Lie Algebras of Linear Forms

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Abstract

Two Lie algebras defined on the Grassman algebra of exterior differential forms are shown to be isomorphic.

The tensorial nature of the Poisson brackets of classical mechanics is "obscured by the accidental fact that on a one-dimensional vector space, there is a one-to-one correspondence between linear forms and numbers, and therefore the derivative at a point is mistaken for a number instead of a linear form" (Dieudonne, 1960). This situation is keenly felt in the Hamiltonian dynamics of holonomic systems. Here the phase space is frequently non-Euclidean, but still having a measure invariant under transformations consistent with the constraints. For example, the phase space of a rigid body under Eulerian rotations is a spherical surface and the 2-form $r^2 \sin\theta \, d\theta \wedge d\varphi$ is the measure invariant under rotations. In such cases, to make the tensorial nature of the Poisson brackets explicit, we will have to extend their definition to include "the fundamental idea of calculus, namely, that the local approximation of functions are linear forms" (Dieudonne, 1960). Such an extension of the Poisson bracket[1] to two linear forms ω and λ is defined by (Shankara, 1967)

$$(\omega, \lambda) = \sum_i (D_{q^i}\omega \wedge D_{p_i}\lambda - D_{p_i}\omega \wedge D_{q^i}\lambda) \tag{1}$$

where the derivatives D_i are ordinary or exterior according as the forms to their right are even or odd. Thus (ω, λ) is always even and the Grassman algebra G of exterior differential forms on the phase space is closed under (1).

It is easy to verify that (1) is bilinear, antisymmetric, and also a derivation over G. Hence

$$V_\omega(\lambda) \equiv (\omega, \lambda) \tag{2}$$

[1] For the quantization of similar systems see Berezin (1975).

is a vector field G' and therefore the commutator on G' satisfies the identity

$$[V_\omega, V_\lambda](\mu) = V_{(\omega,\lambda)}(\mu) \tag{3}$$

for every $\omega, \lambda, \mu \in G$. That is, (1) satisfies the Jacobi identity. Therefore (1) is a Lie bracket on G.

The other natural Lie bracket on G is the commutator

$$[\omega, \lambda] = \omega \wedge \lambda - \lambda \wedge \omega \tag{4}$$

and a study of the mapping of the Lie algebras defined by (1) and (4) brings out some interesting features. In this note we prove the following theorem:

> *Theorem.* For every (ordered) pair of linearly independent odd forms there exists a one-to-one correspondence between their Poisson brackets and the commutators.

Proof. Using the identities

$$(\omega_1 \wedge \omega_2, \lambda) = (\omega_1, \lambda) \wedge \omega_2 + \omega_1 \wedge (\omega_2, \lambda) \tag{5}$$

$$(\omega, \lambda_1 \wedge \lambda_2) = (\omega, \lambda_1) \wedge \lambda_2 + \lambda_1 \wedge (\omega, \lambda_2) \tag{6}$$

the bracket $(\omega_1 \wedge \omega_2, \lambda_1 \wedge \lambda_2)$ can be expanded in two different ways, viz., using (5) and (6) in that order and in the opposite order. Equating these two expressions we get

$$\{(\omega_1, \lambda_1) \wedge \lambda_2 + \lambda_1 \wedge (\omega_1, \lambda_2)\} \wedge \omega_2 + \omega_1 \wedge \{(\omega_2, \lambda_1) \wedge \lambda_2 + \lambda_1 \wedge (\omega_2, \lambda_2)\}$$
$$= \{(\omega_1, \lambda_1) \wedge \omega_2 + \omega_1 \wedge (\omega_2, \lambda_1)\} \wedge \lambda_2 + \lambda_1 \wedge \{(\omega_1, \lambda_2) \wedge \omega_2 + \omega_1 \wedge (\omega_2, \lambda_2)\}$$

i.e.,

$$(\omega_1, \lambda_1) \wedge (\omega_2 \wedge \lambda_2 - \lambda_2 \wedge \omega_2) = (\omega_1 \wedge \lambda_1 - \lambda_1 \wedge \omega_1) \wedge (\omega_2, \lambda_2) \tag{7}$$

provided $\omega_1 \wedge \lambda_1 - \lambda_1 \wedge \omega_1 \neq 0$, $\omega_2 \wedge \lambda_2 - \lambda_2 \wedge \omega_2 \neq 0$. This means that the theorem is restricted to linearly independent odd forms. With this restriction a solution of (7) is given by

$$\sum_i dq^i \wedge dp_i \wedge [\omega, \lambda] = (\omega, \lambda) \tag{8}$$

where the left side is multiplied by the invariant 2-form on the phase space to preserve the degrees on either side without changing their values.

Now (8) defines an isomorphism since the invariant 2-form is isomorphic to R^n and may be chosen as identity. Therefore if we also show that for an (ordered) pair of forms both brackets are unique, the proof will be complete.

Let ω, λ and ω', λ' be two pairs of odd forms. Then $[\omega, \lambda] = [\omega' \lambda']$ implies $\omega \wedge \lambda = \omega' \wedge \lambda'$. If they are also linearly independent as required by the theorem we should have

$$\omega' = A\omega, \qquad \lambda' = A^{-1}\lambda$$

where A is a constant. That is, if the commutators of two pairs of linearly independent odd forms are equal, the forms themselves are equal (up to a constant).

Similarly

$$(\omega, \lambda) = (\omega', \lambda') \Rightarrow V_{(\omega, \lambda)} = V_{(\omega', \lambda')}$$
$$\Rightarrow [V_\omega, V_\lambda] = [V_{\omega'}, V_{\lambda'}] \Rightarrow V_{\omega'} = aV = V_{a\omega}$$

and

$$V_{\lambda'} = a^{-1} V_\lambda = V_{a^{-1}\lambda} \Rightarrow \omega' = a\omega, \lambda' = a^{-1}\lambda$$

where a is a constant. That is, if the Poisson brackets of two pairs of linearly independent odd forms are equal, the forms themselves are equal (up to a constant).

References

Berezin, F. A. (1975). *Communications in Mathematical Physics*, **40**, 153.
Dieudonne, J. (1960). *"Foundations of Modern Analysis"*. (Academic Press, New York).
Shankara, T. S. (1967). *Nuovo Cimento*, 47, 553.